

NUMERICAL RANGES IN II_1 FACTORS

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ABSTRACT. In this paper, we generalize the notion of the C -numerical range of a matrix to operators in arbitrary tracial von Neumann algebras. For each self-adjoint operator C , the C -numerical range of such an operator is defined; it is a compact, convex subset of \mathbb{C} . We explicitly describe the C -numerical ranges of several operators and classes of operators.

1. INTRODUCTION

An interesting invariant of an operator is its numerical range. Given a Hilbert space \mathcal{H} and a bounded linear operator $T : \mathcal{H} \rightarrow \mathcal{H}$, the *numerical range* of T is the set of complex numbers

$$W_1(T) = \{\langle T\xi, \xi \rangle_{\mathcal{H}} \mid \xi \in \mathcal{H}, \|\xi\|_{\mathcal{H}} = 1\}.$$

The Hausdorff-Toeplitz Theorem (see [19, 33]) states that the numerical range of an operator is always a convex subset. Furthermore, when restricting to finite dimensional \mathcal{H} , the numerical range of a matrix is compact and can be used to obtain several interesting structural results, such as that a matrix of trace zero is always unitarily equivalent to a matrix with zeros along the diagonal.

The numerical range of a matrix is often substantially larger than the spectrum and yields cruder information about the matrix. For example, if N is a normal matrix, then $W_1(N)$ is the convex hull of the eigenvalues of N . Therefore, precise information about the eigenvalues of N cannot be obtained from $W_1(N)$.

In [17], Paul Halmos proposed a generalization of the numerical range of a matrix. For each $\xi \in \mathbb{C}^n$ with $\|\xi\|_2 = 1$ and $T \in \mathcal{M}_n(\mathbb{C})$, we have

$$\langle T\xi, \xi \rangle_{\mathbb{C}^n} = \text{Tr}(TP_{\xi})$$

where Tr is the (unnormalized) trace and $P_{\xi} \in \mathcal{M}_n(\mathbb{C})$ is the rank one projection onto $\mathbb{C}\xi$. Thus, for $T \in \mathcal{M}_n(\mathbb{C})$ and $k \in \{1, \dots, n\}$, the k -numerical range of T defined as

$$W_k(T) = \left\{ \frac{1}{k} \text{Tr}(TP) \mid P \in \mathcal{M}_n(\mathbb{C}) \text{ a projection of rank } k \right\}.$$

C. A. Berger showed, using the Hausdorff-Toeplitz Theorem and the fact that $W_1(T)$ is convex, that each $W_k(T)$ is a convex set (see [17, Solution 211]). Operators' k -numerical ranges have been extensively studied and much is known. For example [14, Theorem 1.2] shows

$$W_k(T) = \frac{1}{k} \{ \text{Tr}(TX) \mid 0 \leq X \leq I_n, \text{Tr}(X) = k \}.$$

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It is clear that the set on the right-hand-side of the above equation is a convex set, yet this did not produce an new proof of Berger's result as [14, Theorem 1.2] relied on of Berger's result. These k -numerical ranges provide substantially more information about a matrix than the numerical range alone. Indeed, if $N \in \mathcal{M}_n(\mathbb{C})$ is a normal matrix with eigenvalues $\{\lambda_j\}_{j=1}^n$ listed according to their multiplicities, then, by [14, Theorem 1.5], the k -numerical range of N is the convex hull of the set

$$\left\{ \frac{1}{k} \sum_{j \in K} \lambda_j \mid J \subseteq \{1, \dots, n\}, |J| = k \right\}.$$

By varying k , these sets provide enough information to determine the eigenvalues of N and, thus, to determine N up to unitary equivalence.

In [35], Westwick analyzed a generalization of the k -numerical ranges of a matrix which was later further generalized by Golberg and Straus in [16]. Given two matrices $C, T \in \mathcal{M}_n(\mathbb{C})$, the C -numerical range of T is defined to be the set

$$W_C(T) = \{\text{Tr}(TU^*CU) \mid U \in \mathcal{M}_n(\mathbb{C}) \text{ a unitary}\}. \quad (1)$$

It is not difficult to see that if $C_k \in \mathcal{M}_n(\mathbb{C})$ is a matrix with $\frac{1}{k}$ along the diagonal precisely k times and zeros elsewhere, then $W_{C_k}(T) = W_k(T)$. Thus, the C -numerical ranges are indeed generalizations of the k -numerical ranges.

Using ideas from [19], Westwick in [35] demonstrated that if $C \in \mathcal{M}_n(\mathbb{C})$ is self-adjoint, then $W_C(T)$ is a convex set. However, Westwick also showed that if $C = \text{diag}(0, 1, i) \in \mathcal{M}_3(\mathbb{C})$, then $W_C(C)$ is not convex. Based on [35] and [16], in [31] Poon gave another proof that the C -numerical ranges are convex for self-adjoint $C \in \mathcal{M}_n(\mathbb{C})$. Poon's work gave an alternate description of the C -numerical range based on a notion of majorization for n -tuples of real numbers. This notion of majorization is the one appearing in a classical theorem of Schur ([32]) and Horn ([22]) characterizing the possible diagonal n -tuples of a self-adjoint matrix based on its eigenvalues.

As the notion of majorization has an analogue in arbitrary tracial von Neumann algebras, the goal of this paper is to examine C -numerical ranges in arbitrary von Neumann algebras. In light of the example of Westwick given above, we will restrict our attention to self-adjoint C . Furthermore, we note that analogues of the k -numerical ranges inside diffuse von Neumann algebras have been previously studied in [1–4]. Consequently, the results contained in this paper are a mixture of generalizations of results from [1–4], new proofs of results in [1–4], and additional results. This paper contains a total of six sections, including this one, and is structured as follows.

Section 2 begins by recalling a notion of majorization for elements of $L^\infty[0, 1]$. The generalization of C -numerical ranges to tracial von Neumann algebras is then obtained by applying majorization to eigenvalue functions of self-adjoint operators. After many basic properties of C -numerical ranges are demonstrated, several important results, such as the fact that C -numerical ranges are independent of the von Neumann algebra under consideration, are obtained. Of importance are the results that C -numerical ranges are always compact, convex sets of \mathbb{C} and, if one restricts to type II_1 factors, one can define C -numerical ranges using the closed unitary orbit of C instead of the notion of majorization. In addition, we demonstrate the C -numerical range of T is continuous in both C and T , and we demonstrate results from [1–4] that follow immediately from this different view.

Section 3 is dedicated to describing the C -numerical ranges of self-adjoint operators via eigenvalue functions. This is particularly important for Section 4 which demonstrates a method for computing C -numerical ranges of operators based on knowledge of C -numerical ranges of self-adjoint operators. This is significant as numerical ranges of matrices are often difficult to compute (see [26] for the 3×3 case).

Section 5 computes α -numerical ranges (i.e. the generalization of the k -numerical range of a matrix) for several operators. Although computing the k -numerical ranges of a matrix is generally a hard task, there are several interesting examples of operators in II_1 factors whose α -numerical ranges can be explicitly described. In particular, we demonstrate the existence of normal and non-normal operators whose α -numerical ranges agree for all α .

Section 6 concludes the paper by examining the relationship between α -numerical ranges and conditional expectations of operators onto subalgebras. In particular, we demonstrate that a scalar λ is in the α -numerical range of an operator T in a II_1 factor if and only if there exists diffuse abelian von Neumann subalgebra \mathcal{A} such that the trace of the spectral projection of the expectation of T onto \mathcal{A} corresponding to the set $\{\lambda\}$ is at least α .

2. DEFINITIONS AND BASIC RESULTS

In this section, we generalize the notion of the C -numerical range of a matrix to tracial von Neumann algebras (for self-adjoint C) thereby obtaining more general numerical ranges than those considered in [1–4]. The C -numerical range of an operator is a compact, convex set defined using a notion of majorization for eigenvalue functions of self-adjoint operators and is described via an equation like equation (1) inside II_1 factors. Many properties of C -numerical ranges will be demonstrated including continuity results and lack of dependence on the von Neumann algebra considered.

Throughout this paper, (\mathfrak{M}, τ) will denote a von Neumann algebra \mathfrak{M} possessing a normal, faithful, tracial state, with τ such a state. We will call such a pair a *tracial von Neumann algebra*. Furthermore, $\text{Proj}(\mathfrak{M})$ will denote the set of projections in \mathfrak{M} and \mathfrak{M}_{sa} will be used to denote the set of self-adjoint elements of \mathfrak{M} .

To begin, we will need a concept whose origin is due to Hardy, Littlewood, and Pólya.

Definition 2.1 (see [18]). Let $f, g \in L^\infty[0, 1]$. It is said that f *majorizes* g , denoted $g \prec f$, if

$$\int_0^t g(x) dx \leq \int_0^t f(x) dx \text{ for all } t \in [0, 1] \quad \text{and} \quad \int_0^1 g(x) dx = \int_0^1 f(x) dx.$$

Note if $g \prec f$ and $h \prec g$, one clearly has $h \prec f$.

To apply the above definition, we desire an analogue of eigenvalues for self-adjoint operators in tracial von Neumann algebras. For this section and the rest of the paper, given an normal operator N in a von Neumann algebra, we will use $1_X(N)$ to denote the spectral projection of N corresponding to a Borel set $X \subseteq \mathbb{C}$.

Definition 2.2. Let (\mathfrak{M}, τ) be a diffuse, tracial von Neumann algebra and let $T \in \mathfrak{M}$ be self-adjoint. The *eigenvalue function* of T is defined for $s \in [0, 1]$ by

$$\lambda_T(s) = \inf\{t \in \mathbb{R} \mid \tau(1_{(t, \infty)}(T)) \leq s\}.$$

It is elementary to verify that the eigenvalue function of T is a bounded, non-increasing, right continuous function from $[0, 1)$ to \mathbb{R} . The following result is seemingly folklore, and a proof may be found in [6, Proposition 2.3].

Proposition 2.3. *Let (\mathfrak{M}, τ) be a diffuse, tracial von Neumann algebra and let $T \in \mathfrak{M}$ be self-adjoint. Then there is a projection-valued measure e_T on $[0, 1)$ valued in \mathfrak{M} such that $\tau(e_T([0, t))) = t$ for every $t \in [0, 1)$ and*

$$T = \int_0^1 \lambda_T(s) de_T(s).$$

In particular $\tau(T) = \int_0^1 \lambda_T(s) ds$.

Remark 2.4. Note the von Neumann algebra generated by $\{e_T([0, t))\}_{t \in [0, 1)}$ is isomorphic to a copy of $L^\infty[0, 1]$ inside \mathfrak{M} in such a way that T corresponds to the L^∞ -function $s \mapsto \lambda_T(s)$ and τ restricts to integration against the Lebesgue measure m .

Using the above definitions, we may now define the main objects of study in this paper.

Definition 2.5. Let (\mathfrak{M}, τ) be a tracial von Neumann algebra and let $C \in \mathfrak{M}_{\text{sa}}$. The C -numerical range of an element $T \in \mathfrak{M}$ is the set

$$V_C(T) := \{\tau(TX) \mid X \in \mathfrak{M}_{\text{sa}}, \lambda_X \prec \lambda_C\}.$$

Remark 2.6. It is not difficult to verify that if (\mathfrak{M}, τ) is a tracial von Neumann algebra, if $T, S \in \mathfrak{M}_{\text{sa}}$ with T positive, and if $\lambda_S \prec \lambda_T$, then S must be positive. In addition, it is not difficult to show that if $P \in \mathfrak{M}$ is a projection with $\tau(P) = \alpha \in [0, 1]$, then

$$\{X \in \mathfrak{M}_{\text{sa}} \mid \lambda_X \prec \lambda_P\} = \{X \in \mathfrak{M} \mid 0 \leq X \leq I_{\mathfrak{M}}, \tau(X) = \alpha\}.$$

In analogy, for $\alpha \in (0, 1]$ and $T \in \mathfrak{M}$, we define the α -numerical range of T to be the set

$$V_\alpha(T) := \frac{1}{\alpha} \{\tau(TX) \mid X \in \mathfrak{M}, 0 \leq X \leq I_{\mathfrak{M}}, \tau(X) = \alpha\}.$$

The α -numerical ranges were originally studied (through a multivariate analogue for commuting n -tuples of self-adjoint operators) in the papers [1–4] and the $\frac{1}{\alpha}$ factor is included so that if $0 < \alpha < \beta \leq 1$ then $V_\beta(T) \subseteq V_\alpha(T)$.

The following contains a collection of important properties of C -numerical ranges that mainly follow from properties of eigenvalue functions contained in [12, 13, 30]. Note for two subsets X, Y of \mathbb{C} and $\omega \in \mathbb{C}$, we define

$$\begin{aligned} \omega X &= \{\omega z \mid z \in X\}, \\ \omega + X &= \{\omega + z \mid z \in X\}, \text{ and} \\ X + Y &= \{z + w \mid z \in X, w \in Y\}. \end{aligned}$$

Proposition 2.7. *Let (\mathfrak{M}, τ) be a tracial von Neumann algebra, let $T, S \in \mathfrak{M}$, and let $C, C_1, C_2 \in \mathfrak{M}_{\text{sa}}$. Then*

- (i) $V_C(T)$ is a convex set for all $T \in \mathfrak{M}$,
- (ii) $V_C(T^*)$ equals the complex conjugate of $V_C(T)$,
- (iii) $V_C(\text{Re}(T)) = \{\text{Re}(z) \mid z \in V_C(T)\}$ and $V_C(\text{Im}(T)) = \{\text{Im}(z) \mid z \in V_C(T)\}$,
- (iv) $V_C(T + S) \subseteq V_C(T) + V_C(S)$,

- (v) $V_C(zI_{\mathfrak{M}} + wT) = z\tau(C) + wV_C(T)$ for all $z, w \in \mathbb{C}$,
- (vi) $V_C(U^*TU) = V_C(T)$ for all unitaries $U \in \mathfrak{M}$,
- (vii) $V_{C_1}(T) \subseteq V_{C_2}(T)$ whenever $C_1 \prec C_2$, and
- (viii) $V_{aC+bI_{\mathfrak{M}}}(T) = aV_C(T) + b\tau(T)$ for all $a, b \in \mathbb{R}$.

Proof. For part (i), notice that if $X_1, X_2 \in \mathfrak{M}_{\text{sa}}$ are such that $\lambda_{X_1}, \lambda_{X_2} \prec \lambda_C$, then

$$\lambda_{tX_1+(1-t)X_2} \prec t\lambda_{X_1} + (1-t)\lambda_{X_2} \prec \lambda_C$$

for all $t \in [0, 1]$ by [13, Lemma 2.5 (ii)], by [13, Theorem 4.4], and by a simple translation argument to assume all three operators are positive. Hence it trivially follows that

$$\{X \in \mathfrak{M}_{\text{sa}} \mid \lambda_X \prec \lambda_C\}$$

is a convex set so $V_C(T)$ is convex (being the image under a linear map of a convex set).

Except for parts (vi) and (viii), the other parts are trivial computations. To see part (vi), note $\lambda_{U^*CU} = \lambda_C$ for all unitaries $U \in \mathfrak{M}$ and all $C \in \mathfrak{M}_{\text{sa}}$. To see part (viii), note it is trivial to verify that $\lambda_{C+bI_{\mathfrak{M}}}(s) = \lambda_C(s) + b$ for all $s \in [0, 1]$. If $a \in \mathbb{R}$ is positive, then $\lambda_{aC}(s) = a\lambda_C(s)$ for all $s \in [0, 1]$. Consequently, if a is positive, then $\lambda_X \prec \lambda_C$ if and only if $\lambda_{aX} \prec \lambda_{aC}$ so the result follows. If $a \in \mathbb{R}$ is negative, then one can verify that $\lambda_{aC}(s) = a\lambda_C(1-s)$ for all but a countable number of $s \in [0, 1]$ where the jump discontinuities of $\lambda_C(s)$ occur. One can again verify in this case that $\lambda_X \prec \lambda_C$ if and only if $\lambda_{aX} \prec \lambda_{aC}$ so the result follows. ■

Our next goal is to show the very useful property that the C -numerical ranges of an operator do not depend on the ambient von Neumann algebra. To do so, we recall the following result.

Theorem 2.8 (see [8]). *Let (\mathfrak{M}, τ) be a tracial von Neumann algebra, let \mathfrak{N} be a von Neumann subalgebra of \mathfrak{M} , and let $E_{\mathfrak{N}} : \mathfrak{M} \rightarrow \mathfrak{N}$ be the trace-preserving conditional expectation of \mathfrak{M} onto \mathfrak{N} . Then $\lambda_{E_{\mathfrak{N}}(X)} \prec \lambda_X$ for all $X \in \mathfrak{M}_{\text{sa}}$.*

Proposition 2.9. *Let (\mathfrak{M}, τ) be a tracial von Neumann algebra and let $C \in \mathfrak{M}_{\text{sa}}$. For $T \in \mathfrak{M}$ let $V_C(T)$ denote the C -numerical range as given in Definition 2.5. Let \mathfrak{N} be a von Neumann subalgebra of \mathfrak{M} such that $T \in \mathfrak{N}$. Then*

$$V_C(T) = \{\tau(TX) \mid X \in \mathfrak{N}_{\text{sa}}, \lambda_X \prec \lambda_C\}. \quad (2)$$

In particular, $V_C(T)$ does not depend on the diffuse tracial von Neumann algebra considered.

Proof. The inclusion \supseteq in (2) is clear. For the reverse inclusion, let $E_{\mathfrak{N}} : \mathfrak{M} \rightarrow \mathfrak{N}$ denote the trace-preserving conditional expectation of \mathfrak{M} onto \mathfrak{N} . If $X \in \mathfrak{M}_{\text{sa}}$ is such that $\lambda_X \prec \lambda_C$, then $E_{\mathfrak{N}}(X) \in \mathfrak{N}$, $\lambda_{E_{\mathfrak{N}}(X)} \prec \lambda_X \prec \lambda_C$ by Theorem 2.8, and

$$\tau(TE_{\mathfrak{N}}(X)) = \tau(E_{\mathfrak{N}}(TX)) = \tau(TX).$$

This proves (2). ■

By Proposition 2.9, we may compute the C -numerical ranges in any tracial von Neumann algebra we like. In particular, as every tracial von Neumann algebra embeds in a trace-preserving way into a type II_1 factor, we may restrict our attention to type II_1 factors when considering C -numerical ranges. By doing so, we will obtain an alternate description of C -numerical ranges that is a direct analogue of equation (1) and produces many corollaries. We begin with the following.

Definition 2.10. Let \mathfrak{A} be an arbitrary C^* -algebra and let $\mathcal{U}(\mathfrak{A})$ denote the unitary group of \mathfrak{A} . For $T \in \mathfrak{A}$, the *unitary orbit of T* is the set

$$\mathcal{U}(T) = \{U^*TU \mid U \in \mathcal{U}(\mathfrak{A})\}$$

and the (norm-)closed unitary orbit of T is the set $\mathcal{O}(T) = \overline{\mathcal{U}(T)}^{\|\cdot\|}$.

Remark 2.11. Notice if $T, S \in \mathfrak{M}$ are self-adjoint operators then $\lambda_T \prec \lambda_S$ and $\lambda_S \prec \lambda_T$ if and only if $\lambda_T(s) = \lambda_S(s)$ for all $s \in [0, 1)$. By Definition 2.2, these are equivalent to T and S having the same spectral distribution. It is well-known that these are all equivalent to $T \in \mathcal{O}(S)$, provided \mathfrak{M} is a type II_1 factor.

Notice that if \mathfrak{A} is a finite dimensional C^* -algebra, then $\mathcal{U}(T) = \mathcal{O}(T)$. In general, $\mathcal{O}(T)$ is the correct object to consider when studying infinite dimensional C^* -algebras. In particular, we will use $\mathcal{O}(T)$ to generalize equation (1) to type II_1 factors. In particular, the work of [16, 31] proves the following result when \mathfrak{M} is a matrix algebra.

Theorem 2.12. *Let (\mathfrak{M}, τ) be a type II_1 factor and let $C \in \mathfrak{M}_{\text{sa}}$. Then for all $T \in \mathfrak{M}$,*

$$V_C(T) = \{\tau(TX) \mid X \in \mathfrak{M}_{\text{sa}}, X \in \mathcal{O}(C)\}.$$

To prove Theorem 2.12, we will need two results. The first is the following connection between majorization of eigenvalue functions and convex hulls of unitary orbits.

Theorem 2.13 (see [5, 7, 20, 21, 23–25]). *Let (\mathfrak{M}, τ) be a factor and let $X, T \in \mathfrak{M}_{\text{sa}}$. Then the following are equivalent:*

- (1) $\lambda_X \prec \lambda_T$.
- (2) $X \in \overline{\text{conv}(\mathcal{U}(T))}^{\|\cdot\|}$.
- (3) $X \in \overline{\text{conv}(\mathcal{U}(T))}^{w^*}$.

The second result required to prove Theorem 2.12 is the following technical result, whose proof is contained in the proof of [9, Theorem 5.3] and follows by simple manipulations of functions.

Proposition 2.14 ([9, Theorem 5.3]). *Let (\mathfrak{M}, τ) be a type II_1 factor and let $A, C \in \mathfrak{M}$ be self-adjoint operators such that $\lambda_A \prec \lambda_C$ and $A \notin \mathcal{O}(C)$. Then there exists a non-zero projection $P \in \mathfrak{M}$ and an $\epsilon > 0$ such that $\lambda_{A+PS} \prec \lambda_C$ for all self-adjoint operators $S \in \mathfrak{M}$ satisfying $\|S\| < \epsilon$, $S = PS = SP$, and $\tau(S) = 0$.*

Proof of Theorem 2.12. Fix $C \in \mathfrak{M}_{\text{sa}}$ and $T \in \mathfrak{M}$. Then

$$\{\tau(TX) \mid X \in \mathfrak{M}_{\text{sa}}, X \in \mathcal{O}(C)\} \subseteq V_C(T)$$

by Remark 2.11 and Definition 2.5.

For the other inclusion, fix $X \in \mathfrak{M}_{\text{sa}}$ with $\lambda_X \prec \lambda_C$ and define

$$Q_{X,C} = \{Y \in \mathfrak{M}_{\text{sa}} \mid \tau(TY) = \tau(TX), \lambda_Y \prec \lambda_C\}.$$

Since the linear map $Z \mapsto \tau(TZ)$ is weak*-continuous, by Theorem 2.13 $Q_{X,C}$ is a non-empty (as $X \in Q_{X,C}$), convex, weak*-compact subset. Hence, by the Krein–Milman Theorem, $Q_{X,C}$ has an extreme point, say A .

We will show $A \in \mathcal{O}(C)$ to complete the proof. To see this, suppose to the contrary that $A \notin \mathcal{O}(C)$. Since $A \in Q_{X,C}$, $\lambda_A \prec \lambda_C$ so by Proposition 2.14 there

exists a non-zero projection $P \in \mathfrak{M}$ and an $\epsilon > 0$ such that $\lambda_{A+S} \prec C$ for all self-adjoint operators $S \in \mathfrak{M}$ with $\|S\| < \epsilon$, $S = PS = SP$, and $\tau(S) = 0$.

Consider the linear map

$$\psi : \{S \in \mathfrak{M}_{\text{sa}} \mid S = PS = SP, \tau(S) = 0\} \rightarrow \mathbb{C}$$

defined by $\psi(S) = \tau(TS)$. By dimension requirements, there exists a $S \in \ker(\psi) \setminus \{0\}$. By scaling, we obtain a non-zero $S \in \mathfrak{M}_{\text{sa}}$ such that $\|S\| < \epsilon$, $S = PS = SP$, $\tau(S) = 0$, and $\tau(TS) = 0$. By construction $A \pm S \in Q_{X,C}$ and since

$$A = \frac{1}{2}(A+S) + \frac{1}{2}(A-S)$$

we obtain a contradiction to the fact that A was an extreme point of $Q_{X,C}$. \blacksquare

With Proposition 2.9 and Theorem 2.12 complete, we obtain several important corollaries. In fact, [1] went to great lengths to obtain a (multivariate) version of the following result, for which our techniques provide a quicker proof.

Corollary 2.15. *Let (\mathfrak{M}, τ) be a type II_1 factor, let $T \in \mathfrak{M}$, and let $\alpha \in (0, 1]$. Then*

$$V_\alpha(T) = \frac{1}{\alpha} \{\tau(TP) \mid P \in \text{Proj}(\mathfrak{M}), \tau(P) = \alpha\}.$$

Corollary 2.16. *Let (\mathfrak{M}, τ) be a tracial von Neumann algebra, let $T \in \mathfrak{M}$, and let $C \in \mathfrak{M}_{\text{sa}}$. Then $V_C(T)$ is a compact set.*

Proof. By Proposition 2.9 we may assume that \mathfrak{M} is a type II_1 factor. Hence Theorem 2.8 implies that

$$V_C(T) = \left\{ \tau(TX) \mid X \in \overline{\text{conv}(\mathcal{U}(T))}^{w*} \right\}.$$

As $\overline{\text{conv}(\mathcal{U}(T))}^{w*}$ is weak*-compact and τ is a weak*-continuous linear functional, we obtain that $V_C(T)$ is compact. \blacksquare

Corollary 2.17. *Let (\mathfrak{M}, τ) be a tracial von Neumann algebra and let $T, C \in \mathfrak{M}_{\text{sa}}$. Then $V_C(T) = V_T(C)$.*

Proof. By Proposition 2.9 we may assume that \mathfrak{M} is a type II_1 factor. As $\mathcal{U}(T)$ is (norm-)dense in $\mathcal{O}(T)$ and $\mathcal{U}(C)$ is (norm-)dense in $\mathcal{O}(C)$, we obtain that

$$\{\tau(TU^*CU) \mid U \in \mathfrak{M}, U \text{ a unitary}\}$$

is dense in both $V_C(T)$ and $V_T(C)$ by Theorem 2.12. Hence $V_C(T) = V_T(C)$ as both sets are compact by Corollary 2.16. \blacksquare

Another important corollary is the continuity of the C -numerical range of T as both C and T vary. For this discussion, recall that for compact subsets X and Y of \mathbb{C} , the Hausdorff distance between X and Y is defined to be

$$d_H(X, Y) = \max \left\{ \sup_{x \in X} \text{dist}(x, Y), \sup_{y \in Y} \text{dist}(y, X) \right\}.$$

Proposition 2.18. *Let (\mathfrak{M}, τ) be a tracial von Neumann algebra and let $T \in \mathfrak{M}$. If $C_1, C_2 \in \mathfrak{M}_{\text{sa}}$, then*

$$d_H(V_{C_1}(T), V_{C_2}(T)) \leq \|T\| \|C_1 - C_2\|.$$

In particular, the map $C \mapsto V_C(T)$ is a continuous map from \mathfrak{M}_{sa} (equipped with the operator norm) to the compact, convex subsets of \mathbb{C} equipped with the Hausdorff distance.

Proof. To begin we may assume that \mathfrak{M} is a type II_1 factor by Proposition 2.9. Note for all $X \in \mathcal{O}(C_1)$ and $\epsilon > 0$ there exists an $X' \in \mathcal{O}(C_2)$ such that

$$\|X - X'\| \leq \epsilon + \|C_1 - C_2\|$$

and thus

$$|\tau(TX) - \tau(TX')| \leq \|T\| \|X - X'\| \leq \|T\| \|C_1 - C_2\| + \epsilon \|T\|.$$

As one may also interchange the roles of C_1 and C_2 , the result follows by Theorem 2.12. \blacksquare

Proposition 2.19. *Let (\mathfrak{M}, τ) be a tracial von Neumann algebra, let $T, S \in \mathfrak{M}$, and let $C \in \mathfrak{M}_{\text{sa}}$. Then*

$$d_H(V_C(T), V_C(S)) \leq \|C\| \|T - S\|.$$

Thus, for any fixed $C \in \mathfrak{M}_{\text{sa}}$, the map $T \mapsto V_C(T)$ is continuous from \mathfrak{M} (equipped with the operator norm) to the compact, convex subsets of \mathbb{C} equipped with the Hausdorff distance.

Proof. To begin we may assume that \mathfrak{M} is a type II_1 factor by Proposition 2.9. For all $X \in \mathcal{O}(C)$, notice

$$|\tau(TX) - \tau(SX)| \leq \|T - S\| \|X\| = \|T - S\| \|C\|.$$

Hence the result follows by Theorem 2.12. \blacksquare

Corollary 2.20. *Let (\mathfrak{M}, τ) be a tracial von Neumann algebra and let $T, S \in \mathfrak{M}$. If T and S are approximately unitarily equivalent, that is $S \in \mathcal{O}(T)$, then $V_C(T) = V_C(S)$ for all $C \in \mathfrak{M}_{\text{sa}}$.*

Proof. The result follows from part (vi) of Proposition 2.7 and Proposition 2.19. \blacksquare

3. C -NUMERICAL RANGES OF SELF-ADJOINT OPERATORS

In this section, we will use eigenvalue functions to describe $V_C(T)$ when $C, T \in \mathfrak{M}_{\text{sa}}$. This will be of use in the subsequent section when developing a method for computing C -numerical ranges of an arbitrary operator T .

To begin our description of $V_C(T)$ for all $C, T \in \mathfrak{M}_{\text{sa}}$, we will assume that C and T are positive operators. From the description of such $V_C(T)$, Proposition 2.7 will yield descriptions of $V_C(T)$ for all $C, T \in \mathfrak{M}_{\text{sa}}$.

Proposition 3.1. *Let (\mathfrak{M}, τ) be a tracial von Neumann algebra and let $T, C \in \mathfrak{M}$ be positive. Then*

$$V_C(T) = \left[\int_0^1 \lambda_T(s) \lambda_C(1-s) ds, \int_0^1 \lambda_T(s) \lambda_C(s) ds \right].$$

Remark 3.2. Note if $T, C \in \mathfrak{M}_{\text{sa}}$ with C positive, then we still have

$$V_C(T) = \left[\int_0^1 \lambda_T(s) \lambda_C(1-s) ds, \int_0^1 \lambda_T(s) \lambda_C(s) ds \right]$$

by Proposition 2.7 and the fact that $\lambda_{aI_{\mathfrak{M}}+T}(s) = a + \lambda_T(s)$ for all $s \in [0, 1)$ and $a \in \mathbb{R}$.

To begin the proof of Proposition 3.1, we note by Remark 2.4 and Proposition 2.9 that we may assume $\mathfrak{M} = L^\infty[0, 1]$ equipped with the trace given by integration against Lebesgue measure m and that $T = \lambda_T$ as a function on $[0, 1]$.

To understand C -numerical ranges inside $L^\infty[0, 1]$, we need to understand which functions have the same eigenvalue functions. This returns us to the work of Hardy, Littlewood, and Pólya.

Definition 3.3 ([18, Section 10.12]). For a real-valued function $f \in L^\infty[0, 1]$, the *non-increasing rearrangement* of f is the function

$$f^*(s) = \inf\{x \mid m(\{t \mid f(t) \geq x\}) \leq s\} \text{ for all } s \in [0, 1].$$

It is not difficult to show that if $f \in L^\infty[0, 1]$, then $\lambda_f = f^*$. Consequently f^* is non-increasing, right-continuous function on $[0, 1)$ that is positive when f is positive. Furthermore, if f is a characteristic function, it is not difficult to see how f^* is a rearrangement of f into a non-increasing function.

We begin the demonstration of Proposition 3.1 with the following.

Lemma 3.4. *Let $f, g \in L^\infty[0, 1]$ be non-increasing, positive, right continuous functions where g is a step function. Then*

$$\int_0^1 f(x)g(x) dx = \sup \left\{ \int_0^1 f(x)h(x) dx \mid h^* = g \right\}.$$

and

$$\int_0^1 f(x)g(1-x) dx = \inf \left\{ \int_0^1 f(x)h(x) dx \mid h^* = g \right\}.$$

Proof. By the assumptions on g , there exists $0 = x_0 < x_1 < \dots < x_n = 1$ and $a_1 > a_2 > \dots > a_n \geq 0$ such that

$$g = \sum_{k=1}^n a_k 1_{[x_{k-1}, x_k]}.$$

Suppose $h \in L^\infty[0, 1]$ is such that $h^* = g$. By the definition of the non-increasing rearrangement (also see Remark 2.11), there exists disjoint Borel subsets $\{X_k\}_{k=1}^n$ of $[0, 1]$ such that $m(\bigcup_{k=1}^n X_k) = 1$, $m(X_k) = x_k - x_{k-1}$ for all k , and

$$h = \sum_{k=1}^n a_k 1_{X_k}.$$

We claim that

$$\int_0^1 f(x)h(x) dx \leq \int_0^1 f(x)g(x) dx.$$

To see this, suppose $h \neq g$. Let $k(h)$ be the smallest index so that

$$m([x_{k(h)-1}, x_{k(h)}) \setminus X_{k(h)}) > 0.$$

By the selection of $k(h)$ and since $m(\bigcup_{k=1}^n X_k) = 1$ and $m(X_k) = x_k - x_{k-1}$, there exists a smallest $k'(h) > k(h)$ so that

$$m([x_{k(h)-1}, x_{k(h)}) \cap X_{k'(h)}) > 0.$$

Furthermore, $m(X_{k(h)} \setminus [x_{k(h)-1}, x_{k(h)})) = m([x_{k(h)-1}, x_{k(h)}) \setminus X_{k(h)}) > 0$ as $m(X_k) = x_k - x_{k-1}$, and

$$X_{k(h)} \setminus [x_{k(h)-1}, x_{k(h)}) \subseteq [x_{k(h)}, 1]$$

by the selection of $k(h)$. Therefore, there exists $Y \subseteq X_{k(h)} \setminus [x_{k(h)-1}, x_{k(h)})$ and $Z \subseteq [x_{k(h)-1}, x_{k(h)}) \cap X_{k'(h)}$ so that

$$m(Y) = m(Z) = \min\{m([x_{k(h)-1}, x_{k(h)}) \setminus X_{k(h)}), m([x_{k(h)-1}, x_{k(h)}) \cap X_{k'(h)})\}.$$

If $X_{k(h)}^1 := Z \cup (X_{k(h)} \setminus Y)$, $X_{k'(h)}^1 := Y \cup (X_{k'(h)} \setminus Z)$, $X_k^1 := X_k$ when $k \neq k(h), k'(h)$, and

$$h_1 = \sum_{k=1}^n a_k 1_{X_k^1},$$

then it is elementary to verify that $(h_1)^* = h^* = g$. Furthermore

$$\begin{aligned} \int_0^1 f(x)(h_1(x) - h(x)) dx &= \int_Z f(x)(a_{k(h)} - a_{k'(h)}) dx + \int_Y f(x)(a_{k'(h)} - a_{k(h)}) dx \\ &= (a_{k(h)} - a_{k'(h)}) \left(\int_Z f(x) dx - \int_Y f(x) dx \right) \geq 0 \end{aligned}$$

since $a_{k(h)} - a_{k'(h)} \geq 0$, $m(Z) = m(Y)$, $\sup(Z) \leq \inf(Y)$, and f is a positive, non-increasing function.

If $h_1 \neq g$, then one can repeat the above arguments with h_1 in place of h where one necessarily has either $k(h_1) > k(h)$ or $k(h_1) = k(h)$ and $k'(h_1) > k'(h)$. As there are a finite number of indices, one eventually constructs $h = h_0, h_1, h_2, \dots, h_{m-1}, h_m = g$ with $(h_j)^* = g$ and

$$\int_0^1 f(x)h_j(x) dx \leq \int_0^1 f(x)h_{j+1}(x) dx$$

for all j . Hence, as $h \in L^\infty[0, 1]$ with $h^* = g$ was arbitrary, we obtain

$$\int_0^1 f(x)g(x) dx = \sup \left\{ \int_0^1 f(x)h(x) dx \mid h^* = g \right\}.$$

The other equation in the statement of the result is proved using similar techniques. \blacksquare

Proof of Proposition 3.1. As remarked above, we may assume $\mathfrak{M} = L^\infty[0, 1]$ and $T = \lambda_T$ under this identification. Since the map $X \mapsto \lambda_X$ is operator-norm to $L^\infty[0, 1]$ -norm continuous, and since $T \mapsto V_C(T)$ and $C \mapsto V_C(T)$ are operator-norm to Hausdorff distance continuous, we may assume without loss of generality that T and C have finite spectrum. Consequently, there exists $0 = x_0 < x_1 < \dots < x_n = 1$, $t_1 \geq t_2 > \dots > t_n \geq 0$, and $c_1 \geq c_2 \geq \dots \geq c_n \geq 0$ such that

$$T = \sum_{k=1}^n a_k 1_{[x_{k-1}, x_k)} \quad \text{and} \quad \lambda_C = \sum_{k=1}^n c_k 1_{[x_{k-1}, x_k)}.$$

As $\lambda_C \in \mathfrak{M}$ and

$$\tau(T\lambda_C) = \int_0^1 \lambda_T(x)\lambda_C(x) dx$$

by definition, we clearly have $\int_0^1 \lambda_T(x)\lambda_C(x) dx \in V_C(T)$. Similarly, letting $f(x) = \lambda_C(1 - x)$, we have $f \in \mathfrak{M}$, $f^* = \lambda_C$, and

$$\tau(Tf) = \int_0^1 \lambda_T(x)\lambda_C(1 - x) dx,$$

we clearly have $\int_0^1 \lambda_T(x) \lambda_C(1-x) dx \in V_C(T)$. Since $V_C(T)$ is a compact, convex subset of \mathbb{R} (as C and T are positive), it suffices to show that

$$\sup(V_C(T)) = \int_0^1 \lambda_T(x) \lambda_C(x) dx \quad \text{and} \quad \inf(V_C(T)) = \int_0^1 \lambda_T(x) \lambda_C(1-x) dx$$

to complete the proof.

Suppose $g \in \mathfrak{M}$ is such that $\lambda_g \prec \lambda_C$ (thus g is positive). We desire to show that $\tau(Tg) \leq \tau(T\lambda_C)$. Let \mathfrak{N} be the von Neumann subalgebra of \mathfrak{M} generated by the projections $\{1_{[x_{k-1}, x_k]}\}_{k=1}^n$ and let $E_{\mathfrak{N}} : \mathfrak{M} \rightarrow \mathfrak{N}$ be the trace-preserving conditional expectation onto \mathfrak{N} . By Theorem 2.8, $h = E_{\mathfrak{N}}(g) \in \mathfrak{N}$ is a positive operator with finite spectrum such that $\lambda_h \prec \lambda_g \prec \lambda_C$ and $\tau(Th) = \tau(Tg)$. Hence it suffices to show $\tau(Tg) \leq \tau(T\lambda_C)$ for all $g \in \mathfrak{M}$ with finite spectrum and $\lambda_g \prec \lambda_C$.

For such a g , we may without loss of generality assume $g = g^*$ by Lemma 3.4. Consequently, we may assume there exists $0 = x'_0 < x'_1 < \dots < x'_m = 1$, $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$, $c'_1 \geq c'_2 \geq \dots \geq c'_n \geq 0$, and $b_1 \geq b_2 \geq \dots \geq b_m \geq 0$ such that

$$T = \sum_{k=1}^m a'_k 1_{[x'_{k-1}, x'_k]}, \quad \lambda_C = \sum_{k=1}^m c'_k 1_{[x'_{k-1}, x'_k]}, \quad \text{and} \quad g = \sum_{k=1}^m b_k 1_{[x'_{k-1}, x'_k]}.$$

Since $g \prec \lambda_C$, we obtain that

$$\sum_{k=1}^q b_k (x'_k - x'_{k-1}) \leq \sum_{k=1}^q c'_k (x'_k - x'_{k-1}) \quad (3)$$

for all q with equality when $q = m$. Therefore, setting $a'_{m+1} = 0$, we have

$$\begin{aligned} \tau(T(\lambda_C - g)) &= \sum_{k=1}^m a'_k (c'_k - b_k) (x'_k - x'_{k-1}) \\ &= \sum_{q=1}^m \sum_{j=1}^q (a'_q - a'_{q+1}) (c'_j - b_j) (x'_j - x'_{j-1}). \end{aligned}$$

Since $a'_q - a'_{q+1} \geq 0$ for all q and $\sum_{j=1}^q (c'_j - b_j) (x'_j - x'_{j-1}) \geq 0$ by (3), we obtain $\tau(T(\lambda_C - g)) \geq 0$ as desired.

The proof that

$$\inf(V_C(T)) = \int_0^1 \lambda_T(x) \lambda_C(1-x) dx$$

follows from similar arguments. ■

4. A METHOD FOR COMPUTING C -NUMERICAL RANGES

In this section, we will use Proposition 3.1 together with some additional arguments to develop a method for computing $V_C(T)$ for general $T \in \mathfrak{M}$. This will enable us to show that if one knows all α -numerical ranges of an operator T , one also knows all C -numerical ranges of T .

Given an operator T , the main idea is to reduce the computation of the C -numerical range of T to the C -numerical ranges of the real parts of rotations of T , which are described in terms of eigenvalue functions by Proposition 3.1. This is motivated by [27] (or see the English translation [28]). To begin, we will require the following functions.

Notation 4.1. For a non-empty, bounded subset $E \subseteq \mathbb{C}$, let

$$\sup(\operatorname{Re}(E)) = \sup\{\operatorname{Re}(z) \mid z \in E\}$$

and define $g_E : [0, 2\pi) \rightarrow \mathbb{R}$ by

$$g_E(\theta) = \sup(\operatorname{Re}(e^{i\theta} E)).$$

Proposition 4.2. *For a non-empty, compact, convex set $K \subseteq \mathbb{C}$, the function g_K completely determines K . In particular*

$$K = \{z \in \mathbb{C} \mid \operatorname{Re}(e^{i\theta} z) \leq g_K(\theta) \text{ for all } \theta \in [0, 2\pi)\}.$$

Proof. Let $\Psi(K)$ denote the set on the right-hand-side of the above equation. Since $g_{w+K}(\theta) = \operatorname{Re}(e^{i\theta} w) + g_K(\theta)$ for all $w \in \mathbb{C}$, we have

$$\Psi(w + K) = w + \Psi(K).$$

Thus, we may assume without loss of generality that $0 \in K$.

By definition, it is clear that $K \subseteq \Psi(K)$. For the other inclusion, suppose $w \in K^c$. By the Hahn-Banach Theorem there is a line separating w from K . This line is the solution set in \mathbb{C} of the equation $\operatorname{Re}(e^{-i\theta} z) = c$ for some $\theta \in [0, 2\pi)$ and some $c \geq 0$. Thus, the line $\operatorname{Re}(z) = c$ separates $e^{i\theta} K$ from $e^{i\theta} w$. Since $0 \in K$, we have that $0 \leq g_K(\theta) < c < \operatorname{Re}(e^{i\theta} w)$ so $w \notin \Psi(K)$. ■

Example 4.3. For $a, b \in \mathbb{R}$ with $a, b > 0$, consider the solid ellipse

$$K = \left\{ x + iy \mid x, y \in \mathbb{R}, \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}.$$

The parametrization of the boundary of K in polar coordinates is defined by the map

$$\theta \mapsto a \cos(\theta) + ib \sin(\theta),$$

and from this it is elementary to verify that

$$g_K(\theta) = \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}.$$

As the C -numerical ranges of an operator are compact, convex subsets of \mathbb{C} , in order to determine them it suffices to describe the functions $g_{V_C(T)}(\theta)$. Furthermore, it suffices to describe $V_C(T)$ for C positive by part (viii) of Proposition 2.7 (otherwise we translate C to be a positive operator C' , compute $V_{C'}(T)$, and then translate back).

Method 4.4. Given a tracial von Neumann algebra (\mathfrak{M}, τ) , $T \in \mathfrak{M}$, and a positive $C \in \mathfrak{M}$, by combining Propositions 3.1 and 4.2 we obtain a method of computing $V_C(T)$, provided we can obtain sufficient information about the distributions of the operators $\operatorname{Re}(e^{i\theta} T)$ for $\theta \in [0, 2\pi)$. Indeed, by Proposition 3.1 (or, more specifically, Remark 3.2), we have

$$g_{V_C(T)}(\theta) = \int_0^1 \lambda_{\operatorname{Re}(e^{i\theta} T)}(s) \lambda_C(s) ds.$$

Thus, Proposition 4.2 implies

$$V_C(T) = \left\{ z \in \mathbb{C} \mid \operatorname{Re}(e^{i\theta} z) \leq \int_0^1 \lambda_{\operatorname{Re}(e^{i\theta} T)}(s) \lambda_C(s) ds \text{ for all } \theta \in [0, 2\pi) \right\}.$$

In particular, the above method works provided we can describe λ_C and $\lambda_{\text{Re}(e^{i\theta}T)}$ for all $\theta \in [0, 2\pi)$. In fact, the following theorem demonstrates it suffices to know λ_C for all C in $\text{Proj}(\mathfrak{M})$.

Theorem 4.5. *Let (\mathfrak{M}, τ) be a tracial von Neumann algebra and let $T \in \mathfrak{M}$. Then $\{(C, V_C(T)) \mid C \in \mathfrak{M}_{\text{sa}}\}$ is determined by $\{(P, V_P(T)) \mid P \in \text{Proj}(\mathfrak{M})\}$. In particular, the C -numerical ranges of an operator are determined by the α -numerical ranges of an operator.*

Proof. By Method 4.4 it suffices to prove the result for $T \in \mathfrak{M}_{\text{sa}}$. Furthermore, by part (viii) of Proposition 2.7 and by Proposition 2.18, it suffices to show that if $C \in \mathfrak{M}$ is positive with a finite spectrum, then $V_C(T)$ is determined by $\{(P, V_P(T)) \mid P \in \text{Proj}(\mathfrak{M})\}$.

As $C \in \mathfrak{M}$ has finite spectrum, there exists pairwise orthogonal projections $\{P_k\}_{k=1}^n \subseteq \mathfrak{M}$ and $c_1 > c_2 > \cdots > c_n \geq 0$ such that

$$C = \sum_{k=1}^n c_k P_k.$$

It is elementary to show that if $x_0 = 0$ and $x_k = x_{k-1} + \tau(P_k)$ for all $k \geq 1$, then

$$\lambda_C = \sum_{k=1}^n c_k 1_{[x_{k-1}, x_k]}.$$

Consequently, by Remark 3.2,

$$V_C(T) = \left[\sum_{k=1}^n c_{n-k+1} \int_{x_{k-1}}^{x_k} \lambda_T(x) dx, \sum_{k=1}^n c_k \int_{x_{k-1}}^{x_k} \lambda_T(x) dx \right].$$

Consequently, if one knows $\int_{x_{k-1}}^{x_k} \lambda_T(x) dx$ for all k , then one knows $V_C(T)$.

We claim that each $\int_{x_{k-1}}^{x_k} \lambda_T(x) dx$ is determined by $\{(P, V_P(T)) \mid P \in \text{Proj}(\mathfrak{M})\}$. Indeed if $Q_m = \sum_{k=1}^m P_k$, then Q_m is a projection with $\tau(Q_m) = \sum_{k=1}^m \tau(P_k) = x_m$ and

$$\int_0^{x_m} \lambda_T(x) dx = \sup(V_{Q_m}(T))$$

by Remark 3.2. Hence

$$\int_{x_{k-1}}^{x_k} \lambda_T(x) dx = \sup(V_{Q_k}(T)) - \sup(V_{Q_{k-1}}(T))$$

for all k thereby completing the proof of the claim. ■

5. FURTHER EXAMPLES

Theorem 4.5 demonstrates the α -numerical ranges determine all C -numerical ranges. In this section, we compute the α -numerical ranges of several operators. Although computing the k -numerical ranges of a matrix is generally a hard task, there are several interesting examples of operators in II_1 factor whose α -numerical ranges can be explicitly described.

We begin by noting the following.

Proposition 5.1. *Let (\mathfrak{M}_1, τ_1) and (\mathfrak{M}_2, τ_2) be tracial von Neumann algebras, let $T_1 \in \mathfrak{M}_1$, and let $T_2 \in \mathfrak{M}_2$. If T_1 and T_2 have the same $*$ -distributions, then $V_\alpha(T_1) = V_\alpha(T_2)$ for all $\alpha \in (0, 1]$.*

Proof. By Proposition 2.9, we may assume, without loss of generality, that $\mathfrak{M}_k = W^*(T_k)$ for $k = 1, 2$. Since T_1 and T_2 have the same $*$ -distributions, there exists a trace-preserving isomorphism of $W^*(T_1)$ and $W^*(T_2)$ that sends T_1 to T_2 . This clearly implies $V_\alpha(T_1) = V_\alpha(T_2)$ for all $\alpha \in (0, 1]$, by Definition 2.5. \blacksquare

Recall from the introduction that the k -numerical range of a normal matrix $N \in \mathcal{M}_n(\mathbb{C})$ with eigenvalues $\{\lambda_j\}_{j=1}^n$ is

$$W_k(N) = \text{conv} \left(\left\{ \frac{1}{k} \sum_{j \in K} \lambda_j \mid J \subseteq \{1, \dots, n\}, |J| = k \right\} \right).$$

The following generalizes this result to normal operators with finite spectrum in a tracial von Neumann algebra.

Proposition 5.2. *Let (\mathfrak{M}, τ) be a tracial von Neumann algebra, let $N \in \mathfrak{M}$ be a normal operator such that $\sigma(N) = \{\lambda_k\}_{k=1}^n$, and let $w_k = \tau(1_{\{\lambda_k\}}(N))$ for each $k \in \{1, \dots, n\}$. Then for each $\alpha \in (0, 1]$, we have*

$$V_\alpha(N) = \left\{ \frac{1}{\alpha} \sum_{k=1}^n \lambda_k t_k \mid 0 \leq t_k \leq w_k, \sum_{k=1}^n t_k = \alpha \right\}.$$

Proof. Using Proposition 5.1, we may without loss of generality assume $\mathfrak{M} = L^\infty[0, 1]$ and

$$N = \sum_{k=1}^n \lambda_k 1_{X_k}$$

where $\{X_k\}_{k=1}^n$ are disjoint Borel measurable sets such that $\bigcup_{k=1}^n X_k = [0, 1]$ and $m(X_k) = w_k$ for all k (m the Lebesgue measure).

Consider the surjection

$$\psi : \{X \subseteq [0, 1] \mid X \text{ Borel}, m(X) = \alpha\} \rightarrow \left\{ (t_1, \dots, t_n) \mid 0 \leq t_k \leq w_k, \sum_{k=1}^n t_k = \alpha \right\}$$

defined by

$$\psi(X) = (m(X \cap X_1), \dots, m(X \cap X_n)).$$

If $X \subseteq [0, 1]$ is Borel measurable with $m(X) = \alpha$, then

$$\tau(N 1_X) = \int_X \sum_{k=1}^n \lambda_k 1_{X_k}(s) ds = \sum_{k=1}^n \lambda_k t_k$$

where $(t_1, \dots, t_n) = \psi(X)$. Since every $P \in \text{Proj}(L^\infty[0, 1])$ is of the form $P = 1_X$ where $X \subseteq [0, 1]$ and $\tau(P) = m(X)$, the result follows, using Corollary 2.15. \blacksquare

For our next example, recall that a Haar unitary is a unitary element whose spectral distribution is Haar measure on the unit circle.

Example 5.3. Let (\mathfrak{M}, τ) be a tracial von Neuman algebra, let $U \in \mathfrak{M}$ be a Haar unitary, and let \mathbb{D} denote the closed unit disk. For every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, λU and U have the same spectral distribution. Therefore, Proposition 5.1 implies

$$V_\alpha(U) = V_\alpha(\lambda U) = \lambda V_\alpha(U)$$

for every $\alpha \in (0, 1]$ and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Since each $V_\alpha(U)$ is a compact, convex set, this implies

$$V_\alpha(U) = r(\alpha) \mathbb{D}$$

where $r : (0, 1] \rightarrow [0, 1]$ is such that $r(\alpha) = \sup\{\text{Re}(z) \mid z \in V_\alpha(U)\} = \sup V_\alpha(\text{Re}(U))$ where the last equality is part (iii) of Proposition 2.7.

To compute $r(\alpha)$, note that by Proposition 5.1 we may assume that $U = (s \mapsto e^{is}) \in L^\infty[-\pi, \pi]$, so $\text{Re}(U) = (s \mapsto \cos(s))$ and, arguing as in the proof of Proposition 3.1, we deduce

$$r(\alpha) = \frac{1}{2\pi\alpha} \int_{-\pi\alpha}^{\pi\alpha} \cos(s) ds = \frac{1}{\pi\alpha} \sin(\pi\alpha).$$

Thus $V_\alpha(U) = \frac{1}{\pi\alpha} \sin(\pi\alpha) \mathbb{D}$ for all $\alpha \in (0, 1]$.

The above example exhibits a method for computing α -numerical ranges, provided there exists sufficient symmetry.

Corollary 5.4. *Let (\mathfrak{M}, τ) be a diffuse tracial von Neumann algebra and suppose $T \in \mathfrak{M}$ is such that*

$$V_\alpha(T) = e^{i\theta} V_\alpha(T) \text{ for all } \theta \in [0, 2\pi).$$

Then $V_\alpha(T)$ is the closed disk centered at the origin of radius $r_\alpha(T)$, where

$$r_\alpha(T) = \frac{1}{\alpha} \int_0^\alpha \lambda_{\text{Re}(T)}(s) ds = \sup V_\alpha(\text{Re}(T)).$$

Of course, the above corollary applies whenever the $*$ -distribution of T is the same as the $*$ -distribution of $e^{i\theta}T$ for all $\theta \in \mathbb{R}$.

Using Method 4.4, we may compute the α -numerical ranges of several interesting operators.

Example 5.5. Consider the infinite tensor view of the hyperfinite II_1 factor

$$\mathfrak{R} = \bigotimes_{n \geq 1} \mathcal{M}_2(\mathbb{C})$$

and consider the Tucci operator [34]

$$T = \sum_{n \geq 1} \frac{1}{2^n} (\underbrace{I_2 \otimes \cdots \otimes I_2}_{n-1 \text{ times}} \otimes Q \otimes I_2 \otimes \cdots)$$

where $Q = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. This operator is quasinilpotent and generates \mathfrak{R} . To compute $V_\alpha(T)$ for every $\alpha \in (0, 1]$, we first notice that T and $e^{i\theta}T$ are approximately unitarily equivalent via the unitaries

$$U_{n,\theta} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \otimes I_2 \otimes I_2 \otimes \cdots,$$

as $U_{n,\theta}^*(e^{i\theta}T)U_{n,\theta}$ approximate T in norm. Therefore, Corollary 2.20 and Corollary 5.4 imply

$$V_\alpha(T) = r_\alpha(T) \mathbb{D}$$

where \mathbb{D} denotes the closed unit disk and $r_\alpha(T)$ may be computed by as

$$r_\alpha(T) = \sup(V_\alpha(\text{Re}(T))).$$

Let

$$A_0 = \text{Re}(Q) = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$\operatorname{Re}(T) = \sum_{n \geq 1} \frac{1}{2^n} (I_2 \otimes \cdots \otimes I_2 \otimes A_0 \otimes I_2 \otimes \cdots).$$

However, since $2A_0$ is unitarily equivalent to

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

we obtain that $\operatorname{Re}(T)$ is approximately unitarily equivalent to

$$S = \frac{1}{2} \sum_{n \geq 1} \frac{1}{2^n} (I_2 \otimes \cdots \otimes I_2 \otimes A \otimes I_2 \otimes \cdots).$$

Thus, Corollary 2.20 implies

$$r_\alpha(T) = \sup(V_\alpha(S)).$$

Notice

$$\sum_{n=1}^2 \frac{1}{2^n} (I_2 \otimes \cdots \otimes I_2 \otimes A \otimes I_2 \otimes \cdots) = \operatorname{diag} \left(\frac{3}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{3}{4} \right).$$

Furthermore

$$\sum_{n=1}^3 \frac{1}{2^n} (I_2 \otimes \cdots \otimes I_2 \otimes A \otimes I_2 \otimes \cdots) = \operatorname{diag} \left(\frac{7}{8}, \frac{5}{8}, \frac{3}{8}, \frac{1}{8}, -\frac{1}{8}, -\frac{3}{8}, -\frac{5}{8}, -\frac{7}{8} \right).$$

This pattern continues and thus we see that the spectral scale of S is

$$\lambda_S(s) = \frac{1}{2}(1 - 2s).$$

Thus,

$$r_\alpha(T) = \frac{1}{2\alpha} \int_0^\alpha (1 - 2s) ds = \frac{1}{2}(1 - \alpha)$$

so

$$V_\alpha(T) = \frac{1}{2}(1 - \alpha)\mathbb{D}.$$

It is not very difficult to construct a normal operator N satisfying $V_\alpha(N) = V_\alpha(T)$ for all $\alpha \in (0, 1]$, namely, having the same numerical ranges as the quasinilpotent operator T . Indeed, considering the radially symmetric distribution ν on the unit disk such that $\nu(r\mathbb{D}) = 1 - \sqrt{1 - r^2}$ for $0 < r < 1$, one can show that the marginal distribution of ν is uniform measure on $[-1, 1]$. It follows that the normal operator N whose trace of spectral measure is ν satisfies $\lambda_{\operatorname{Re}(N)}(s) = \frac{1}{2}(1 - 2s)$ for all $s \in [0, 1]$ and this implies $V_\alpha(N) = V_\alpha(T)$ for all $\alpha \in (0, 1]$.

Example 5.6. Recall a $(0, 1)$ -circular operator is an element Z of a tracial von Neumann algebra of the form

$$Z = \frac{1}{\sqrt{2}}(X + iY),$$

where X and Y are freely independent $(0, 1)$ -semicircular operators. As the $*$ -distribution of Z is the same as the $*$ -distribution of $e^{i\theta}Z$ for all $\theta \in \mathbb{R}$, Corollary 5.4 implies

$$V_\alpha(Z) = r_\alpha(Z)\mathbb{D}$$

where $r_\alpha(Z) = \sup(V_\alpha(\text{Re}(Z)))$. Since the spectral distribution of $\text{Re}(Z) = \frac{1}{\sqrt{2}}X$ is given by the semicircular law

$$\frac{1}{\pi} 1_{[-\sqrt{2}, \sqrt{2}]}(x) \sqrt{2 - x^2},$$

we obtain that

$$r_\alpha(Z) = \frac{1}{\pi} \int_{h(\alpha)}^{\sqrt{2}} x \sqrt{2 - x^2} dx = \frac{1}{3\pi\alpha} (2 - h(\alpha)^2)^{3/2},$$

where $h(\alpha) \in [-\sqrt{2}, \sqrt{2})$ is such that

$$\frac{1}{\pi} \int_{h(\alpha)}^{\sqrt{2}} \sqrt{2 - x^2} dx = \alpha.$$

Thus, h is the inverse with respect to composition of the decreasing function $f : [-\sqrt{2}, \sqrt{2}] \rightarrow [0, 1]$ given by

$$f(y) = \frac{1}{\pi} \int_y^{\sqrt{2}} \sqrt{2 - x^2} dx = \frac{1}{2} - \frac{1}{2\pi} y \sqrt{2 - y^2} - \frac{1}{\pi} \arcsin\left(\frac{y}{\sqrt{2}}\right).$$

We note the asymptotic expansions

$$f(\sqrt{2} - x) = \frac{2^{7/4}}{3\pi} x^{3/2} - \frac{1}{5\pi 2^{3/4}} x^{5/2} + O(x^{7/2}) \quad (\text{as } x \rightarrow 0^+),$$

$$h(\alpha) = \sqrt{2} - \frac{(3\pi)^{2/3}}{2^{7/6}} \alpha^{2/3} - \frac{(3\pi)^{4/3}}{5(2^{23/6})} \alpha^{4/3} + O(\alpha^2) \quad (\text{as } \alpha \rightarrow 0^+),$$

$$r_\alpha(Z) = \sqrt{2} - \frac{3^{5/3} \pi^{2/3}}{5(2^{7/6})} \alpha^{2/3} + O(\alpha) \quad (\text{as } \alpha \rightarrow 0^+).$$

For comparison, a $(0, 1)$ -circular element has norm 2 and spectrum equal to the disk centred at the origin of radius 1. Note that, since the push-forward measure of the spectral distribution of the normalized Lebesgue measure on the disk of radius $\sqrt{2}$ onto the real axis produces the semicircular law $\frac{1}{\sqrt{2}}X$, Z is an easy example of a non-normal operator such that there exists a normal operator N with $V_\alpha(Z) = V_\alpha(N)$ for all $\alpha \in (0, 1]$.

Example 5.7. The quasinilpotent DT-operator S was introduced in [10] as one of an interesting class of operators in the free group factor $L(\mathbb{F}_2)$, that can be realized as limits of upper triangular random matrices. As the name suggests, its spectrum is $\{0\}$, and it satisfies $\|S\| = \sqrt{e}$ and $\tau(S^*S) = 1/2$. In [11], it was shown that S generates $L(\mathbb{F}_2)$ and that S has many non-trivial hyperinvariant subspaces. Moreover, $\text{Re}(S) = \frac{1}{2}X$, where X is a $(0, 1)$ -semicircular operator and the $*$ -distribution of S is the same as that of $e^{i\theta}S$ for all $\theta \in \mathbb{R}$. Thus, the method of Corollary 5.4 applies, exactly as in Example 5.6, to yield

$$V_\alpha(S) = r_\alpha(S)\mathbb{D},$$

where $r_\alpha(S) = \frac{1}{\sqrt{2}}r_\alpha(Z)$, where $r_\alpha(Z)$ is the function as defined in Example 5.6. Note that the normal measure whose distribution is uniform measure on the disk of radius $\frac{1}{\sqrt{2}}$ has the same α -numerical ranges as the quasinilpotent operator S .

Example 5.8. As a generalization of Example 5.6, consider the operator

$$T = \cos(\psi)X + i \sin(\psi)Y$$

where $\psi \in (0, \frac{\pi}{2})$ and X and Y are freely independent $(0, 1)$ -semicircular operators. In particular, the case $\psi = \frac{\pi}{4}$ produces the circular operator studied in Example 5.6. These elliptic variants of circular operators were studied by Larsen in [29], where he showed

- $\|T\| = 2$,
- the spectrum of T is $\left\{ z \in \mathbb{C} \mid \frac{\operatorname{Re}(z)^2}{\cos^4(\psi)} + \frac{\operatorname{Im}(z)^2}{\sin^4(\psi)} \leq 4 \right\}$, and
- the Brown measure of T is uniform distribution on its spectrum.

To determine $V_\alpha(T)$, we apply Method 4.4. Note that $\operatorname{Re}(e^{i\theta}T)$ is

$$\cos(\psi) \cos(\theta)X - \sin(\psi) \sin(\theta)Y,$$

which is $(0, b(\theta)^2)$ -semicircular where

$$b(\theta) = \sqrt{\cos^2(\psi) \cos^2(\theta) + \sin^2(\psi) \sin^2(\theta)}.$$

Thus the spectral distribution of $\operatorname{Re}(e^{i\theta}T)$ is the same as the spectral distribution of $\sqrt{2}b(\theta)\operatorname{Re}(Z)$, where Z is the $(0, 1)$ -circular operator from Example 5.6. Hence

$$g_{V_\alpha(T)}(\theta) = \sqrt{2}r_\alpha(Z)b(\theta).$$

Therefore, by Proposition 4.2 and Example 4.3, we find

$$V_\alpha(T) = \left\{ z \in \mathbb{C} \mid \frac{\operatorname{Re}(z)^2}{\cos^2(\psi)} + \frac{\operatorname{Im}(z)^2}{\sin^2(\psi)} \leq 2r_\alpha(Z)^2 \right\}.$$

It is curious, although not surprising, that the eccentricity of the ellipse bounding $V_\alpha(T)$ is (except in the circular case $\psi = \frac{\pi}{4}$) different from the eccentricity of the ellipse bounding the spectrum $\sigma(T)$.

To complete this section, we note the following interpolation result that generalizes [15, Corollary 1]. This enables one to obtain knowledge pertaining to one α -numerical range based on others. We note that further results in [15] also have immediate generalizations to α -numerical ranges.

Proposition 5.9. *Let (\mathfrak{M}, τ) be a diffuse, tracial von Neumann algebra and let $T \in \mathfrak{M}$. If $0 < \alpha < \beta < \gamma \leq 1$, then*

$$\frac{\alpha(\gamma - \beta)}{\beta(\gamma - \alpha)}V_\alpha(T) + \frac{\gamma(\beta - \alpha)}{\beta(\gamma - \alpha)}V_\gamma(T) \subseteq V_\beta(T).$$

Proof. Let $\lambda \in V_\alpha(T)$ and let $\mu \in V_\gamma(T)$. By definition, there exist positive contractions $X, Y \in \mathfrak{M}$ such that $\tau(X) = \alpha$, $\tau(Y) = \gamma$,

$$\lambda = \frac{1}{\alpha}\tau(TX), \quad \text{and} \quad \mu = \frac{1}{\gamma}\tau(TY).$$

Let

$$Z = \frac{\gamma - \beta}{\gamma - \alpha}X + \frac{\beta - \alpha}{\gamma - \alpha}Y \in \mathfrak{M}.$$

It is clear that Z is a positive operator such that

$$Z \leq \frac{\gamma - \beta}{\gamma - \alpha}I_{\mathfrak{M}} + \frac{\beta - \alpha}{\gamma - \alpha}I_{\mathfrak{M}} = I_{\mathfrak{M}}$$

and

$$\tau(Z) = \frac{\gamma - \beta}{\gamma - \alpha} \alpha + \frac{\beta - \alpha}{\gamma - \alpha} \gamma = \beta.$$

Finally,

$$\frac{\alpha(\gamma - \beta)}{\beta(\gamma - \alpha)} \lambda + \frac{\gamma(\beta - \alpha)}{\beta(\gamma - \alpha)} \mu = \frac{1}{\beta} \frac{\gamma - \beta}{\gamma - \alpha} \tau(TX) + \frac{1}{\beta} \frac{\beta - \alpha}{\gamma - \alpha} \tau(TY) = \frac{1}{\beta} \tau(TZ) \in V_\beta(T),$$

completing the proof. \blacksquare

Remark 5.10. One may ask whether set equality must occur in Proposition 5.9. Taking $T \in \mathfrak{M}$ to be a Haar unitary, Example 5.3 implies this question asks (by letting $\gamma = 1$) whether

$$\frac{1 - \beta}{\pi(\beta - \alpha\beta)} \sin(\pi\alpha) \mathbb{D} + 0 = \frac{1}{\pi\beta} \sin(\pi\beta) \mathbb{D}$$

holds for all $0 < \alpha < \beta < 1$. As this is clearly not the case, equality need not occur in Proposition 5.9. However, one may use [3] to demonstrate that equality occurs in Proposition 5.9 when T is an $n \times n$ matrix, $\alpha = \frac{k}{n}$, and $\gamma = \frac{k+1}{n}$ for some $k \in \{1, \dots, n\}$.

6. NUMERICAL RANGES AND DIAGONALS

In this our final section, we desire description of when a scalar belongs to the α -numerical range of an operator based on the possible ‘diagonals’ of an operator. Our characterization is similar to that for k -numerical ranges of matrices found in [14, Theorem 2.4]. Unfortunately, we do not obtain true ‘diagonals’ as we do not know if one can guarantee \mathcal{A} in the following technical lemma (whose proof is a generalization of a matricial result) is a MASA.

Lemma 6.1. *Let (\mathfrak{M}, τ) be a type II_1 factor and let $T \in \mathfrak{M}$ be such that $\tau(T) = 0$. Then there exists a diffuse abelian von Neumann subalgebra \mathcal{A} of \mathfrak{M} such that $E_{\mathcal{A}}(T) = 0$, where $E_{\mathcal{A}} : \mathfrak{M} \rightarrow \mathcal{A}$ is the normal conditional expectation.*

Proof. Notice $0 \in V_1(T) \subseteq V_{\frac{1}{2}}(T)$. Hence there exists a projection $P \in \mathfrak{M}$ such that $\tau(P) = \frac{1}{2}$ and $\tau(TP) = 0$. Similarly, $\tau(T(I_{\mathfrak{M}} - P)) = 0$. By repeating this argument in $P\mathfrak{M}P$ and $(I_{\mathfrak{M}} - P)\mathfrak{M}(I_{\mathfrak{M}} - P)$, we obtain four projections $\{P_k\}_{k=1}^4$ such that P_k commutes with P and $I_{\mathfrak{M}} - P$, $\tau(P_k) = \frac{1}{4}$, and $\tau(TP_k) = 0$ for all k . By continuing to repeat the first argument on each compression and by taking the von Neumann algebra generated by these projections, the desired diffuse abelian von Neumann subalgebra of \mathfrak{M} is obtained. \blacksquare

Proposition 6.2. *Let (\mathfrak{M}, τ) be a type II_1 factor, let $T \in \mathfrak{M}$, and let $\alpha \in (0, 1]$. Then $\lambda \in V_\alpha(T)$ if and only if there exists a diffuse abelian von Neumann subalgebra \mathcal{A} of \mathfrak{M} such that $\tau(1_{\{\lambda\}}(E_{\mathcal{A}}(T))) \geq \alpha$, where $E_{\mathcal{A}} : \mathfrak{M} \rightarrow \mathcal{A}$ is the normal conditional expectation.*

Proof. Suppose \mathcal{A} a diffuse abelian von Neumann subalgebra of \mathfrak{M} such that $\beta := \tau(1_{\{\lambda\}}(E_{\mathcal{A}}(T))) \geq \alpha$. Thus

$$\lambda = \tau(E_{\mathcal{A}}(T) 1_{\{\lambda\}}(E_{\mathcal{A}}(T))) = \tau(T 1_{\{\lambda\}}(E_{\mathcal{A}}(T))) \in V_\beta(E_{\mathcal{A}}(T)) \subseteq V_\alpha(E_{\mathcal{A}}(T)).$$

(See Remark 2.6.)

For the converse direction, suppose $\lambda \in V_\alpha(T)$. By part (v) of Proposition 2.7, we may without loss of generality assume that $\lambda = 0$. Since $0 \in V_\alpha(T)$, by

Corollary 2.15 there exists a projection P of trace α such that $\frac{1}{\alpha}\tau(TP) = 0$. Hence $\tau_{P\mathfrak{M}P}(PTP) = 0$ where $\tau_{P\mathfrak{M}P}$ is the trace for $P\mathfrak{M}P$. By Lemma 6.1 there exists a diffuse abelian von Neumann subalgebra \mathcal{A}_0 of $P\mathfrak{M}P$ such that $E_{\mathcal{A}_0}(PTP) = 0$. If \mathcal{A}' is any diffuse abelian von Neumann subalgebra of $(I_{\mathfrak{M}} - P)\mathfrak{M}(I_{\mathfrak{M}} - P)$, then $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}' \subseteq \mathfrak{M}$ is a diffuse abelian von Neumann subalgebra containing P such that $E_{\mathcal{A}}(T)P = 0$. Hence $\tau(1_{\{\lambda\}}(E_{\mathcal{A}}(T))) \geq \alpha$ as desired. ■

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